

Hawking Radiation of Topological Massive Warped-AdS₃ Black Hole Families

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Abstract We investigate the Dirac particles tunnelling as a radiation of Warped AdS₃ black hole family in Topological Massive Gravity. Using the Hamilton-Jacobi method, we discuss tunnelling probability and Hawking temperature of the spin-1/2 particles for the black hole and its extremal cases. We observe that the Hawking temperature of the non-extremal black hole higher than the extremal black hole when $\omega < \frac{2r_0}{3}$, because the non-extremal black hole become unstable in this case.

1 Introduction

A self-consistent quantum gravity theory hasn't been constructed yet. Therefore, the quantum mechanical properties of a classical gravitational field is studied by the quantum mechanical behaviour of a physical system effected from it. In particular, thanks to the extension of standard Quantum theory to curved spacetime, some events, such as particle creation and thermal radiation of a black hole, can be predicted. Moreover, the black holes as the most popular concepts of the classical gravity are just understood by the quantum mechanical concepts. From this point of view, the solutions of the relativistic quantum mechanical wave equations in a gravitational background became an important tool for getting information about its nature [1, 2, 3]. For this reason, the relativistic quantum mechanical wave equations in a curved spacetime background have been extensively studied [4, 5, 6, 7].

The nature of black holes has been understood by thermodynamical and quantum mechanical concepts since 1970 [8, 9, 10, 11, 12, 13]. Among these concepts, thermal

radiation, known as Hawking radiation in the literature, has been investigated as a quantum tunnelling effect of the relativistic particles from a black hole [14, 15, 16, 17, 18, 19, 20, 21, 22, 23]. Thanks to the studies, a black hole temperature, also called hawking temperature in the literature, is related to the black hole's surface gravity. Therefore, the Hawking temperature becomes an important concept to investigate the black hole physics. Since then, in the framework of the standard Einstein general relativity, the Hawking radiation as a tunnelling process of the particles from various black holes has been studied extensively in the literature in both 3+1 and 2+1 dimensional spacetimes [24, 25, 26, 27, 32, 33, 35]. On the other hand, Kerner and Mann extended the tunnelling process to include the Dirac particle emission from a 3+1 dimensional black hole [22, 23]. Also, Ren and Li considered the Dirac particles' tunnelling process to investigate the Hawking radiation for the 2+1-dimensional BTZ black hole by using the tunneling method [27]. The particle tunnelling process in all these studies give useful information about the mathematical and physical properties of the black holes. In a similar way, the Hawking radiation is used to discuss the properties of the black hole in the context of modified gravitation theories [28, 29, 30, 31]. As an example, Gecim and Sucu discussed Hawking radiations for both Dirac and scalar particles from the New-type black hole in the framework of 2+1 dimensional New Massive Gravity theory [33]. However, according to the method, both particles probe the black hole in same way. Also, in the context of modified gravitation theories, Qi investigated the fermion tunnelling radiation from the static Lifshitz black hole in 2+1 dimensional New Massive Gravity theory, and from New Class Black Holes in 3+1 Einstein-Gauss-Bonnet Gravity [35].

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The (2+1) dimensional gravitational models provide a suitable area to investigate the quantum effects of the gravity [36,37,38,39]. Among these, Topologically Massive Gravity as an interesting modified three-dimensional gravitation theory is formed by adding a Chern-Simons term to the standard Einstein-Hilbert action [40]. With this term, the gravity theory has gained both physically and mathematically interesting properties. However, in contrast to other gravitational theories, the graviton becomes a massive particle [41,42,43,44].

The warped AdS_3 black hole for the solution of the Topological massive gravity is given by the following metric [45].

$$ds^2 = N(r)^2 dt^2 - \frac{1}{N(r)^2 F(r)^2} dr^2 - F(r)^2 [d\phi + N^\phi(r) dt]^2 \quad (1)$$

The abbreviations used in here are as follows;

$$F(r)^2 = r^2 + 4\omega r + 3\omega^2 + \frac{r_0^2}{3}$$

$$N(r)^2 = \frac{r^2 - r_0^2}{F(r)^2}, N^\phi(r) = -\frac{2r + 3\omega}{F(r)^2}.$$

The Warped- AdS_3 Black holes have two horizons at $r = \mp r_0$. The parameters ω and r_0 are related to mass and angular momentum of the black hole [45]. The Warped- AdS_3 Black hole becomes extremal at $r_0 = 0$. Additionally, in the extremal case, the black hole has a double horizon at $r = 0$. Moreover, this result does not depend on parameter ω . In an even more special case ($\omega = r_0 = 0$), the metric (1) is reduced to the horizonless metric that is characterized as ground state or ‘vacuum’ of the black-hole [45].

For the metric, the surface gravity is calculated by classical (standard) method as,

$$\kappa = \frac{1}{2} \left[F(r) \frac{\partial}{\partial r} [N^2(r)] \right]_{r=r_0} \quad (2)$$

and thus,

$$\kappa = \sqrt{3} \left(\frac{r_0}{2r_0 + 3\omega} \right).$$

The Hawking temperature, T_H , defined in terms of the surface gravity is $T_H = \frac{\hbar\kappa}{2\pi}$ and, for the black hole, it is given as follows

$$T_H = \frac{\hbar\sqrt{3}}{2\pi} \left(\frac{r_0}{2r_0 + 3\omega} \right).$$

In all studies in the literature, for the non-extremal case, the Hawking temperature is worked out by the surface gravity method that coincides with Hawking temperature obtained by quantum mechanical tunnelling method. In the tunnelling method, the Cauchy integral

has a first order (simple) pole in the horizon of black hole [14,15,16,17,18,19,20,21,22,23,24,25,26,27,28,29,30,31,32,33,35]. Therefore, the surface gravity method is agreed with tunnelling method in the case that the Cauchy integral has a simple pole. On the other hand, according to (2), the surface gravity becomes zero in the extremal black hole, hence the Hawking temperature of the extremal black hole is zero. However, in the context of tunnelling method, as we will show in the following section (Section 3) in this study, the relativistic quantum mechanical particle can be tunnelled from the singularity of the extremal black hole.

The organization of this work is as follows. In the Section 2, we write the Dirac equation in Warped- AdS_3 Black holes background, and calculate the tunnelling possibility of the Dirac particle by using the semi-classical method. Also, we find Hawking temperature. In the Section 3, we carry out the same calculation for the extremal case, $r_0 = 0$ (and $\omega \neq 0$), and for the ground state ($\omega = r_0 = 0$) of the black hole. Finally, we evaluate and summarize the results. In this study, we use $G_N = k_B = c = 1$.

2 Tunnelling of Dirac particles from the Warped- AdS_3 Black Hole

To understand the quantum mechanical properties of the black hole, we find the probability of tunnelling and Hawking temperature by using the solution of the relativistic quantum mechanical wave equation for the Dirac particles. To investigate the tunnelling of the Dirac particles from Warped- AdS_3 Black hole, we write Dirac equation in (2+1) dimensional spacetime in the following representation [46],

$$\{i\bar{\sigma}^\mu(x) [\partial_\mu - \Gamma_\mu(x)]\} \Psi(x) = \frac{m_0}{\hbar} \Psi(x). \quad (3)$$

In this representation; Dirac spinor, $\Psi(x)$, has only two components corresponding positive and negative energy eigenstates that has only one spin polarization. $\bar{\sigma}^\mu(x)$ are the spacetime dependent Dirac matrices and they are written in terms of constant Dirac matrices, $\bar{\sigma}^i$, by using triads, $e_{(i)}^\mu(x)$, as follows

$$\bar{\sigma}^\mu(x) = e_{(i)}^\mu(x) \bar{\sigma}^i, \quad (4)$$

where $\bar{\sigma}^i$ are Dirac matrices in a flat spacetime and given as

$$\bar{\sigma}^i = (\bar{\sigma}^0, \bar{\sigma}^1, \bar{\sigma}^2), \quad (5)$$

with

$$\bar{\sigma}^0 = \sigma^3, \bar{\sigma}^1 = i\sigma^1, \bar{\sigma}^2 = i\sigma^2, \quad (6)$$

where σ^1 , σ^2 and σ^3 Pauli matrices, and $\Gamma_\mu(x)$ are the spin affine connection by the following definition,

$$\Gamma_\mu(x) = \frac{1}{4} g_{\lambda\alpha} (e_{\nu,\mu}^i e_i^\alpha - \Gamma_{\nu\mu}^\alpha) s^{\lambda\nu}(x). \quad (7)$$

Here, $\Gamma_{\nu\mu}^\alpha$ is Christoffel symbol, and $g_{\mu\nu}(x)$ is the space-time dependent metric tensor and given in term of triads as follows,

$$g_{\mu\nu}(x) = e_\mu^{(i)}(x) e_\nu^{(j)}(x) \eta_{(i)(j)}, \quad (8)$$

where μ and ν are curved spacetime indices running from 0 to 2. i and j are flat spacetime indices running from 0 to 2 and $\eta_{(i)(j)}$ is the metric of (2+1) dimensional Minkowski spacetime, with signature (1,-1,-1), and $s^{\lambda\nu}(x)$ is a spin operator given by

$$s^{\lambda\nu}(x) = \frac{1}{2} [\bar{\sigma}^\lambda(x), \bar{\sigma}^\nu(x)]. \quad (9)$$

From Eq.(1) and (8), the triads of $e_{(i)}^\alpha$ are written as;

$$\begin{aligned} e_{(0)}^\mu &= \left(\frac{1}{N}, 0, -\frac{N^\phi}{N} \right), \\ e_{(1)}^\mu &= (0, FN, 0), \\ e_{(2)}^\mu &= \left(0, 0, \frac{1}{F} \right). \end{aligned}$$

The tunnelling probability for the classically forbidden trajectory from inside to outside of the black hole horizon is given by

$$\Gamma = e^{-\frac{2}{\hbar} ImS} \quad (10)$$

where S is the classical action function of a particle trajectory [22,27,47,48,49,50]. Therefore, in order to discuss tunneling probability, one needs to calculate the imaginary part of the classical action function, S , in regards to the tunnelling probability [15]. To investigate the tunnelling probability of a Dirac particle from the black hole, we use the following ansatz for the wave function in the Eq.(3);

$$\Psi(x) = \exp\left(\frac{i}{\hbar} S(t, r, \phi)\right) \begin{pmatrix} A(t, r, \phi) \\ B(t, r, \phi) \end{pmatrix} \quad (11)$$

where $A(t, r, \phi)$ and $B(t, r, \phi)$ are functions of space-time [27,47]. To apply the Hamilton-Jacobi method, we insert the Eq.(11) in the Dirac equation given by Eq.(3). Dividing by the exponential term and neglecting the terms with \hbar , we derive the following two coupled differential equations.

$$\begin{aligned} & A \left[m_0 N(r) + \frac{\partial S}{\partial t} - N^\phi(r) \frac{\partial S}{\partial \phi} \right] \\ & + B \left[i F(r) N(r)^2 \frac{\partial S}{\partial r} + \frac{N(r)}{F(r)} \frac{\partial S}{\partial \phi} \right] = 0 \\ & A \left[i F(r) N(r)^2 \frac{\partial S}{\partial r} - \frac{N(r)}{F(r)} \frac{\partial S}{\partial \phi} \right] \\ & + B \left[m_0 N(r) - \frac{\partial S}{\partial t} + N^\phi(r) \frac{\partial S}{\partial \phi} \right] = 0. \end{aligned} \quad (12)$$

These two equations have nontrivial solutions for $A(t, r, \phi)$ and $B(t, r, \phi)$ when the determinant of the coefficient matrix is vanished. Accordingly,

$$\begin{aligned} & F(r) \left(\frac{\partial S}{\partial t} \right)^2 - 2F(r)^2 N^\phi(r) \left(\frac{\partial S}{\partial t} \right) \left(\frac{\partial S}{\partial \phi} \right) \\ & + \left(F(r)^2 N^\phi(r)^2 - N(r)^2 \right) \left(\frac{\partial S}{\partial \phi} \right)^2 \\ & - N(r)^4 F(r)^4 \left(\frac{\partial S}{\partial r} \right)^2 - N(r)^2 F(r)^2 m_0^2 = 0. \end{aligned} \quad (13)$$

As (∂_t) and (∂_ϕ) are two killing vectors, we can separate $S(t, r, \phi)$ to the variables as follows

$$S(t, r, \phi) = -Et + j\phi + K(r) + C, \quad (14)$$

where E and j are the energy and angular momentum of a Dirac particle, respectively, and C is a complex constant. Inserting Eq.(14) in Eq.(13) and solving for the radial function, $K(r)$, for fixed $\phi = \phi_0$ we get

$$K_\pm(r) = \pm \int \frac{\sqrt{F(r)^2 E^2 - F(r)^2 N(r)^2 m_0^2}}{F(r)^2 N(r)^2} dr. \quad (15)$$

Because of the tunnelling event occurring at outer horizon, the Eq.(15) can be taken as a counter integral around the horizon to calculate the imaginary part [15]. To do this, near the outer horizon, we can expand $f(r) = F(r)^2 N(r)^2$ as

$$f(r_0) = (r - r_0) \left(\frac{df(r)}{dr} \right) + \frac{1}{2} (r - r_0)^2 \left(\frac{d^2 f(r)}{dr^2} \right) + \dots \quad (16)$$

Choosing the lowest-order term in Eq.(16), the integration in Eq.(15) around the simple pole $r = r_0$ can be calculated with help of residue theorem. So we get,

$$K_\pm(r) = \pm i\pi \sqrt{3} E (2r_0 + 3\omega) \quad (17)$$

where $K_+(r)$ is outgoing and $K_-(r)$ is incoming solutions of radial part. The total imaginary part of the action is $ImS(t, r, \phi) = ImK_\pm(r) = ImK_+(r) - ImK_-(r)$ [33,52]. Hence, the two kind probabilities of crossing the outer horizon, from outside to inside or from inside to outside, are given by

$$\begin{aligned} P_{out} &= \exp \left[-\frac{2}{\hbar} ImK_+(r) \right] \\ P_{in} &= \exp \left[-\frac{2}{\hbar} ImK_-(r) \right]. \end{aligned} \quad (18)$$

From the Eq.(17), we find that $ImK_+(r) = -ImK_-(r)$. And, the tunnelling probability of the Dirac particle from the outer event horizon is given by [22,47,51],

$$\begin{aligned} \Gamma &= \frac{P_{out}}{P_{in}} \\ &= \exp \left[-\frac{2\pi E (2r_0 + 3\omega)}{\hbar \sqrt{3} r_0} \right]. \end{aligned} \quad (19)$$

If one expands the classical action in terms of the particle energy, the Hawking temperature is obtained at the lowest order (linear order). So, we can write

$$\Gamma = e^{-\frac{2}{\hbar}ImS} = e^{-\beta E} \quad (20)$$

where β is the inverse temperature of the outer horizon. Where, the Hawking temperature is given as follows

$$T_H = \frac{\hbar\sqrt{3}}{2\pi} \left(\frac{r_0}{2r_0 + 3\omega} \right) \quad (21)$$

This result is consistent with the result of the classical gravity. According to this relation, the Hawking temperature increases as ω gets smaller values of r_0 . However, the angular momentum of the black hole becomes increasingly negative in case $\omega < \frac{\sqrt{5}r_0}{3}$ [45]. For which the black hole emit gravitational radiation and extract angular momentum from the system [56, 57, 58]. It means that the system lost its energy and momentum when $\omega < \frac{\sqrt{5}r_0}{3}$. Therefore, the system is unstable.

3 Tunnelling of the Particles from the Extremal Warped-AdS₃ Black Hole

Extremal black hole solutions have an important role in the black-hole thermodynamics. The common idea in the literature is that the Hawking temperature of an extremal black hole vanishes because the surface gravity is zero according to the classical method. However, extremal black holes make radiation when they have charge [54, 55]. On the other hand, the third law of black-hole dynamics, the analogy between the thermodynamics and the Black hole dynamics laws, point out that the surface gravity of a black hole can not reach to zero [53]. With this motivation, we want to study the tunnelling event in the extremal Warped-AdS₃ black hole by using the Hamilton-Jacobi method.

Warped-AdS₃ Black hole becomes extremal in $r_0 = 0$ ($\omega \neq 0$). In this case, the black hole has a double horizon at $r = 0$. So the coefficients in the Eq.(1) are reduced to following abbreviations.

$$F(r)^2 = r^2 + 4\omega r + 3\omega^2, \quad N(r)^2 = \frac{r^2}{F(r)^2}, \quad N^\phi(r) = -\frac{2r + 3\omega}{F(r)^2}. \quad (22)$$

In the context of tunnelling method, inserting the Eq.(22) and Eq.(14) in the Eq.(13) for fixed $\phi = \phi_0$, the radial part of the action function becomes as follows,

$$K_\pm(r) = \pm \int \frac{\sqrt{E^2(r^2 + 4\omega r + 3\omega^2) - r^2 m_0^2}}{r^2} dr. \quad (23)$$

To calculate the counter integral in Eq.(23), we can use an appropriate $i\epsilon$ prescription to specify the complex contour over which the integral has to be performed

around $r = 0$. In this prescription, the singularity at $r = 0$ shifted to $r = \pm i\epsilon$ where the upper sign should be chosen for the outgoing particles and the lower sign should be chosen for the ingoing particles [15]. Hence, the action for the outgoing particles with help of the contour in the upper complex plane,

$$S_{em} = \lim_{\epsilon \rightarrow 0} \int_{-\epsilon}^{+\epsilon} \frac{\sqrt{E^2(3\omega^2 + 4\omega(r-i\epsilon) + (r-i\epsilon)^2) - m_0^2(r-i\epsilon)^2}}{(r-i\epsilon)^2} dr + (real\ part) \quad (24)$$

With respect to the Residue theorem and using the Eq.(16), the Eq.(24) counter integral around the singularity located at $r = i\epsilon$ has a second order pole and, hence becomes,

$$S_{em} = (real\ part) + i \frac{2\pi E}{\sqrt{3}}.$$

Similarly, the actions for the ingoing particles with the help of the contour in the lower complex plane,

$$S_{ab} = -\lim_{\epsilon \rightarrow 0} \int_{+\epsilon}^{-\epsilon} \frac{\sqrt{E^2(3\omega^2 + 4\omega(r+i\epsilon) + (r+i\epsilon)^2) - m_0^2(r+i\epsilon)^2}}{(r+i\epsilon)^2} dr + (real\ part) \quad (25)$$

and hence,

$$S_{ab} = (real\ part) - i \frac{2\pi E}{\sqrt{3}}.$$

According to Eq.(10), the real part of the action does not contribute to tunnelling probability [15]. Hence, the Dirac particle tunnelling probability and Hawking temperature are obtained as follows, respectively,

$$\Gamma = \exp \left[-\frac{8\pi E}{\hbar\sqrt{3}} \right]$$

and

$$T_H = \frac{\hbar\sqrt{3}}{8\pi}, \quad (26)$$

where all of the contribution to the tunnelling probability and the temperature stem from the second order pole of the complex integral in Hamilton-Jacobi method. As we have shown above, according to Hamilton-Jacobi method, the particles could be tunnelled from the singularity. The Hawking temperature of this radiation becomes a constant and, as the result of this, the extremal black hole has a surface gravity of $\kappa = \frac{\sqrt{3}}{4}$. These results are in agreement in regards to the analogy between the thermodynamics laws and the Black hole dynamics laws [53].

Based on the Eq.(21) and Eq.(26), it is seen that the temperature of extremal black hole is higher than the temperature of non-extremal black hole, for all cases where $\omega > \frac{2r_0}{3}$. However, in the $\omega < \frac{2r_0}{3}$ cases, it seems that the temperature of non-extremal black hole is higher than the temperature of extremal black hole.

On the other hand, the super angular momentum of the non-extremal black hole becomes increasingly negative for $\omega < \frac{2}{3} \frac{r_0}{3}$ [45]. Therefore, in this case the black hole is unstable. Hence, according to Chandrasekhar-Friedman-Schutz (*CFS*) mechanism [56,57], the non-extremal black hole losses its angular momentum and energy via gravitational radiation [58]. However, the gravitational radiation gives a contribution to the Hawking temperature. Therefore, this situation can be reasonable for the non-extremal black hole Hawking temperature higher than the extremal black hole Hawking temperature.

For the vacuum state ($\omega = r_0 = 0$), the coefficients of the metric given by Eq.(1) reduced to following abbreviations [45]:

$$F(r)^2 = r^2, \quad N(r)^2 = 1, \quad N^\phi(r) = -\frac{2}{r}. \quad (27)$$

If we apply the Hamilton-Jacobi method to this case, we find two different Hawking temperatures corresponding to angular part of the action of the tunnelling particle. Firstly, if we take arbitrary angle, i.e. arbitrary ϕ , and with help of the Eq.(14), Eq.(15) and Eq.(27), the tunnelling probability and the Hawking temperature of the Dirac particle can be obtained as,

$$\Gamma = \exp \left[-\frac{8\pi E}{\hbar\sqrt{3}} \right]$$

and

$$T_H = \frac{\hbar\sqrt{3}}{8\pi},$$

which is the same result as of the $r_0 = 0$ extremal case, because the counter integral has a second order pole in the singularity $r = 0$. Secondly, when we consider a fixed angle, i.e. $\phi = \phi_0$, the counter integral which obtained by using the Eq.(14), Eq.(15) and Eq.(27) gives the following equation for $W(r)$ (the radial part):

$$W = \pm \int \frac{\tilde{E}}{r} dr$$

where $\tilde{E} = \sqrt{E^2 - m_0^2}$. This counter integral has a simple pole in the singularity $r = 0$. Hence, the tunnelling probability and the Hawking temperature of the Dirac particle are calculated as follow respectively,

$$\Gamma = \exp \left[-\frac{4\pi E}{\hbar} \right]$$

and

$$T_H = \frac{\hbar}{4\pi}.$$

These results shown that the temperature of the vacuum state is higher than other states.

Under the framework of findings obtained in this study, one can say that the thermodynamical properties of the extremal case is not a limit for thermodynamical properties the non-extremal case.

4 Conclusions

In this study, we have studied Hawking radiation of Dirac particles as a quantum tunnelling effect from the Warped-AdS₃ Black holes. By using Hamilton-Jacobi method, we have derived the tunneling probability of the relativistic particles from the Warped-AdS₃ Black holes. Subsequently, using the obtained these particle tunnelling probabilities, we have calculated the Hawking temperature for the black hole. These results are consistent with surface gravity method based on previous works in the literature.

We have also examined the particle tunneling whether it is possible for the extremal cases of the black hole or not. Our results show that Dirac particles may radiate from the extremal Warped-AdS₃ black holes.

We can summarize our results as follow.

- ($r_0 = 0$ and $\omega \neq 0$): Hawking temperature of the radiation from extremal Warped-AdS₃ black hole is $T_H = \frac{\hbar\sqrt{3}}{8\pi}$. We infer from the quantum mechanical result that the extremal black hole has a surface gravity which is, classically, not predicted. From the Hawking temperature and surface gravity relation, we get the surface gravity, as $\kappa = \frac{\sqrt{3}}{4}$. The κ expression can be interpreted as quantized ground state surface gravity of the extremal Warped-AdS₃ black hole.
- ($r_0 = 0$ and $\omega = 0$): This case represents the vacuum state of the Warped-AdS₃ black hole. If we take an arbitrary angle, the tunnelling probability and the Hawking temperature of the Dirac particle can be obtained as in Case-1. But, if we take a fixed angle, $\phi = \phi_0$, the tunnelling probability and the Hawking temperature of the Dirac particle calculated as $\Gamma = \exp \left[-\frac{4\pi E}{\hbar} \right]$ and $T_H = \frac{\hbar}{4\pi}$, respectively.
- According to *CFS* mechanism, in the $\omega < \frac{2}{3} \frac{r_0}{3}$ case the non-extremal black hole losses its angular momentum and energy because of gravitational radiation. Hence the Hawking temperature of the non-extremal black hole in the $\omega < \frac{2}{3} \frac{r_0}{3}$ case, becomes higher than the Hawking temperature of the extremal black hole because gravitational radiation give contribution to Hawking radiation because of the instability. On the other hand, temperature of extremal black hole is higher than the temperature of non-extremal black hole, for all cases where $\omega > \frac{2}{3} \frac{r_0}{3}$.

All of these results show that the classical surface gravity is in accordance with Hawking temperature calculated from the imaginary part of the complex integral with the first order pole in the Hamilton-Jacobi method. The classical surface gravity becomes zero in that case the complex integral pole is of second order,

but, quantum mechanically, the particles keep tunneling from the extremal black holes, i.e. the extremal black hole has a Hawking temperature and thus has a surface gravity. Also, although the Hamilton-Jacobi method predicts a surface gravity for an extremal black hole, this method, unfortunately, terminates the spin effect of the particles to the results. Therefore, it is obtained the same results for scalar particle and particles with spin [23,33].

Using the Hamilton-Jacobi tunnelling method for the extremal Reissner-Nordström (RN) black hole, one can see that the Hawking radiation is different zero ($T_H = \frac{\hbar}{4\pi M}$) in the charged case, but the Hawking radiation vanishes in the neutral case. As shown in this study, the rotating extremal black holes have Hawking radiation as in the charged extremal black holes [54, 55].

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